THE FATOU THEOREM AND ITS CONVERSE

BY

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1. **Introduction.** Let H^+ denote the class of functions which are non-negative and harmonic in the upper half plane $y>0$ and let H denote the class of functions which can be expressed as the difference of two functions in H^+; obviously $H^+ \subset H$. It is well known that a function $u(x, y)$ has the Poisson-Stieltjes representation

$$(1.1) \qquad u(x, y) = Ky + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(s-x)^2+y^2}\, d\alpha(s),$$

where K is a constant and

$$(1.2) \qquad \int_{-\infty}^{\infty} \frac{|\, d\alpha(s)\,|}{s^2+1} < \infty,$$

if and only if $u(x, y)$ is in H (see [5]). Hence with each function $u(x, y)$ in H we can associate a function $\alpha(s)$ and a constant K such that (1.1) and (1.2) hold; we can further assume that $\alpha(0)=0$. For each $u(x, y)$ in H^+, $\alpha(s)$ is nondecreasing and $K \geq 0$.

In this paper we are concerned with the behaviour of a function $u(x, y)$, in H, and its derivatives as (x, y) approaches a point P on the x-axis. For convenience we take P to be the origin.

Our starting point is the following pair of results (see [4]).

FATOU THEOREM. *Suppose that $u(x, y)$ is in H and that $\alpha'(0)=A$. Then for each $0<\theta<\pi$, $u(r\cos\theta, r\sin\theta)\to A$ as $r\to 0+$.*

The converse of this theorem is not true unless we restrict ourselves to the subclass H^+. In this case we have the following result.

LOOMIS THEOREM. *Suppose that $u(x, y)$ is in H^+, that $0<a\neq b<\pi$, and that for $\theta=a$ and $\theta=b$,*

$$u(r\cos\theta, r\sin\theta) \to A$$

as $r\to 0+$. Then $\alpha'(0)=A$.

The Fatou theorem is an Abelian theorem, the Loomis theorem is the corresponding Tauberian theorem, and the Tauberian condition is the restriction that $u(x, y)\geq 0$.

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In §3 we prove several extensions of the Fatou theorem and in §4 we present the corresponding converses. The arguments in §4 are very simple and make use of a variant of the Wiener Tauberian theorem which is developed in §2. Finally in §5 we consider an extension of a well known result due to Lindelöf.

2. **A Tauberian theorem.** In this section we introduce a two dimensional form of the Wiener Tauberian theorem for Stieltjes integrals. We begin with some definitions and notation.

A function $f(x)$ is in L if it is Lebesgue integrable over the real line. For $f(x)$ in L we let

$$F = \int_{-\infty}^{\infty} f(x)dx, \qquad F(y) = \int_{-\infty}^{\infty} f(x)e^{ixy}dx \tag{2.1}$$

for all real y.

A function $f(x)$ is in M if it is continuous for all x and if

$$\sum_{n=-\infty}^{\infty} \operatorname*{Max}_{n\leq x\leq n+1} |f(x)| < \infty. \tag{2.2}$$

Obviously $M \subset L$.

A function $\beta(x)$ is in V if it has bounded variation over each finite interval and if

$$\int_{n}^{n+1} |d\beta(x)| \tag{2.3}$$

is bounded in n.

LEMMA 1. *Suppose that $f(x)$ and $g(x)$ are in L. Then $k(x) = f * g(x)$, that is*

$$k(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy,$$

exists p.p. in x, $k(x)$ is in L, and $K(y) = F(y)G(y)$. Furthermore $k(x)$ is in M $f(x)$ and/or $g(x)$ is in M.

Proof. The first part of this lemma follows from known results. For the last part assume that $f(x)$ is in M. Then the fact that $f(x)$ is bounded and continuous allows us to apply the Lebesgue "dominated convergence" theorem to conclude that $k(x)$ is continuous. Next we see that

$$\sum_{n=-\infty}^{\infty} \operatorname*{Max}_{n\leq x\leq n+1} |k(x)| \leq \sum_{n=-\infty}^{\infty} \operatorname*{Max}_{n\leq x\leq n+1} \int_{-\infty}^{\infty} |f(x-y)| \, |g(y)| \, dy$$
$$\leq \sum_{n=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} \operatorname*{Max}_{n-m-1\leq x\leq n-m+1} |f(x)| \int_{m}^{m+1} |g(y)| \, dy \right\}$$

and, inverting the order of summation, we conclude that

$$\sum_{n=-\infty}^{\infty} \operatorname*{Max}_{n \le x \le n+1} |k(x)| \le \left(\int_{-\infty}^{\infty} |g(y)|\,dy\right)\left(\sum_{n=-\infty}^{\infty} \operatorname*{Max}_{n \le x \le n+2} |f(x)|\right) < \infty$$

which completes the argument.

The Wiener Tauberian Theorem for Stieltjes integrals can be stated, using (2.1), as follows (see [3, p. 294]).

WIENER THEOREM. *Suppose that $f(x)$ is in M, that $F(y) \neq 0$ for all y, that $\beta(x)$ is in V, and that*

$$\int_{-\infty}^{\infty} f(x-y)\,d\beta(y) \to B$$

as $x \to +\infty$. Then for each $h(x)$ in M,

$$\int_{-\infty}^{\infty} h(x-y)\,d\beta(y) \to \frac{B}{F} H$$

as $x \to +\infty$.

With the aid of Lemma 1 we obtain the following result.

THEOREM 1. *Suppose that $f_1(x), f_2(x), g_1(x), g_2(x)$ are in M, that $F_1(y)G_2(y) - F_2(y)G_1(y) \neq 0$ for all y, that $\beta(x)$, $\gamma(x)$ are in V, and that*

$$\tag{2.4} \begin{aligned} \int_{-\infty}^{\infty} f_1(x-y)\,d\beta(y) + \int_{-\infty}^{\infty} f_2(x-y)\,d\gamma(y) \to B, \\ \int_{-\infty}^{\infty} g_1(x-y)\,d\beta(y) + \int_{-\infty}^{\infty} g_2(x-y)\,d\gamma(y) \to C \end{aligned}$$

as $x \to +\infty\ (-\infty)$. Then for each $h(x)$ in M

$$\tag{2.5} \begin{aligned} \int_{-\infty}^{\infty} h(x-y)\,d\beta(y) \to \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1} H, \\ \int_{-\infty}^{\infty} h(x-y)\,d\gamma(y) \to \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1} H \end{aligned}$$

as $x \to +\infty\ (-\infty)$.

Proof. We consider the case where $x \to +\infty$; the case where $x \to -\infty$ then follows with trivial modifications. Set

$$b(x) = \int_{-\infty}^{\infty} f_1(x-y)\,d\beta(y) + \int_{-\infty}^{\infty} f_2(x-y)\,d\gamma(y),$$

$$c(x) = \int_{-\infty}^{\infty} g_1(x-y)\,d\beta(y) + \int_{-\infty}^{\infty} g_2(x-y)\,d\gamma(y).$$

Both $b(x)$ and $c(x)$ are bounded by virtue of the hypotheses, (2.2), and (2.3). Hence with (2.4) and the Lebesgue theorem we conclude that

$$\begin{aligned} b*g_2(x) &= \int_{-\infty}^{\infty} b(x-y)g_2(y)dy \to BG_2, \\ c*f_2(x) &= \int_{-\infty}^{\infty} c(x-y)f_2(y)dy \to CF_2 \end{aligned} \tag{2.6}$$

as $x \to +\infty$. Moreover, from the Fubini theorem it follows that

$$\begin{aligned} b*g_2(x) &= \int_{-\infty}^{\infty} f_1*g_2(x-y)d\beta(y) + \int_{-\infty}^{\infty} f_2*g_2(x-y)d\gamma(y), \\ c*f_2(x) &= \int_{-\infty}^{\infty} g_1*f_2(x-y)d\beta(y) + \int_{-\infty}^{\infty} g_2*f_2(x-y)d\gamma(y) \end{aligned} \tag{2.7}$$

and, with $k(x) = f_1*g_2(x) - f_2*g_1(x)$, we conclude from (2.6) and (2.7) that

$$\int_{-\infty}^{\infty} k(x-y)d\beta(y) \to BG_2 - CF_2 \tag{2.8}$$

as $x \to +\infty$. By Lemma 1, $k(x)$ is in M,

$$K(y) = F_1(y)G_2(y) - F_2(y)G_1(y) \neq 0$$

for all y, and we can apply the Wiener theorem to (2.8) to obtain the first part of (2.5). The second part follows in exactly the same way.

To obtain a more convenient form for our Tauberian theorem we perform a familiar exponential change of variable and shift from the interval $-\infty < x < \infty$ to the interval $0 < t < \infty$. (For the details see [3, pp. 295–296].) The analogues for L, M, and V are as follows.

A function $f(t)$ is in L' if it is Lebesgue integrable over $0 < t < \infty$. For $f(t)$ in L' we let

$$F = \int_0^{\infty} f(t)dt, \qquad F(y) = \int_0^{\infty} f(t)t^{iy}dt$$

for all real y.

A function $f(t)$ is in M' if $f(t)$ is continuous over $0 < t < \infty$ and if

$$\sum_{n=-\infty}^{\infty} \operatorname*{Max}_{e^n \le t \le e^{n+1}} |tf(t)| < \infty.$$

Again $M' \subset L'$. Observe that $f(t)$ is in M' if $f(t)$ is continuous over $0 \le t < \infty$ and if, for some $\epsilon > 0$,

$$f(t) = O(t^{-1-\epsilon}) \tag{2.9}$$

for large t (see [3, p. 299]).

A function $\beta(t)$ is in V' if it has bounded variation over each finite interval in $0<t<\infty$ and if

$$\int_t^{et} \frac{|d\beta(s)|}{s}$$

is bounded for all $t>0$.

THEOREM 2. *Suppose that $f_1(t), f_2(t), g_1(t), g_2(t)$ are in M', that $F_1(y)G_2(y) - F_2(y)G_1(y) \neq 0$ for all y, that $\beta(t)$, $\gamma(t)$ are in V', and that*

$$\frac{1}{t}\int_0^\infty f_1\left(\frac{s}{t}\right)d\beta(s) + \frac{1}{t}\int_0^\infty f_2\left(\frac{s}{t}\right)d\gamma(s) \to B,$$

$$\frac{1}{t}\int_0^\infty g_1\left(\frac{s}{t}\right)d\beta(s) + \frac{1}{t}\int_0^\infty g_2\left(\frac{s}{t}\right)d\gamma(s) \to C$$

as $t\to+\infty\,(0+)$. Then for each $h(t)$ in M',

$$\text{(2.10)}\qquad \begin{aligned} &\frac{1}{t}\int_0^\infty h\left(\frac{s}{t}\right)d\beta(s) \to \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1}H, \\ &\frac{1}{t}\int_0^\infty h\left(\frac{s}{t}\right)d\gamma(s) \to \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1}H \end{aligned}$$

as $t\to+\infty\,(0+)$.

We require the following corollary in §4.

COROLLARY 1. *Suppose, in addition to the hypotheses of Theorem 2, that $\beta(t)$ and $\gamma(t)$ are monotone and that $\beta(0)=\gamma(0)=0$. Then*

$$\text{(2.11)}\qquad \frac{\beta(t)}{t} \to \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1}, \quad \frac{\gamma(t)}{t} \to \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1}$$

as $t\to+\infty\,(0+)$.

Proof. If we set $h(t)=e^{-t}$, then $h(t)$ is in M' and we conclude, from (2.10), that

$$\frac{1}{t}\int_0^\infty e^{-(s/t)}d\beta(s) \to \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1},$$

$$\frac{1}{t}\int_0^\infty e^{-(s/t)}d\gamma(s) \to \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1}$$

as $t\to+\infty\,(0+)$. Now (2.11) follows from a familiar Tauberian theorem for the Laplace transform. (See [8, p. 192].)

3. **Extensions of the Fatou theorem.** Here we develop some Abelian results related to the Fatou theorem. It will be convenient to make use of the

complex notation, writing $u(x+iy)$ for $u(x, y)$ and letting $R(w)$ and $I(w)$ denote the real and imaginary parts, respectively, of the complex number w.

For $u(z)$, harmonic, and $n \geq 1$ we let

$$D_\psi^n u = \cos\psi \frac{\partial^n u}{\partial x^n} + \sin\psi \frac{\partial^n u}{\partial y \partial x^{n-1}}$$

denote the nth directional derivative of $u(z)$ in the direction ψ. Observe that

$$D_\psi^n u = R\left(e^{i\psi} \frac{d^n f}{dz^n}\right),$$

where $f(z)$ is analytic with $u = R(f)$, and that any linear combination of nth derivatives of $u(z)$ can be expressed as $CD_\psi u(z)$, where C is a constant and ψ is an appropriate angle.

For $u(z)$ in H we have the Poisson-Stieltjes representation (1.1) which, in complex notation, reduces to

$$u(z) = Ky + \frac{1}{\pi}\int_{-\infty}^{\infty} I\left\{\frac{1}{s-z}\right\} d\alpha(s). \tag{3.1}$$

The Lebesgue theorem and (1.2) allow us to differentiate underneath the integral sign in (3.1) to obtain

$$D_\psi^1 u(z) = K\sin\psi + \frac{1}{\pi}\int_{-\infty}^{\infty} I\left\{\frac{e^{i\psi}}{(s-z)^2}\right\} d\alpha(s) \tag{3.2}$$

and, for $n > 1$,

$$D_\psi^n u(z) = \frac{n!}{\pi}\int_{-\infty}^{\infty} I\left\{\frac{e^{i\psi}}{(s-z)^{n+1}}\right\} d\alpha(s). \tag{3.3}$$

Next for any $c > 0$, it follows from (1.2) that

$$\int_{|s|\geq c} \frac{|d\alpha(s)|}{|s-z|^2} = O(1)$$

as $z \to 0$ and hence

$$\begin{gathered} \int_{|s|\geq c} I\left\{\frac{1}{s-z}\right\} d\alpha(s) = O(y), \\ \int_{|s|\geq c} I\left\{\frac{e^{i\psi}}{(s-z)^{n+1}}\right\} d\alpha(s) = O(1) \end{gathered} \tag{3.4}$$

as $z \to 0$.

We list some formulas which are required later on.

LEMMA 2[1]. *Suppose that* $-1<R(\zeta)<1$, *that* $0<\theta<\pi$, *and that* $n\geq 1$. *Then*

$$\frac{1}{\pi}\int_0^\infty I\left\{\frac{1}{s-e^{i\theta}}\right\}s^\zeta ds=\frac{\sin(\pi-\theta)\zeta}{\sin\pi\zeta},\tag{3.5}$$

$$-\frac{1}{\pi}\int_0^\infty I\left\{\frac{1}{s+e^{i\theta}}\right\}s^\zeta ds=\frac{\sin\theta\zeta}{\sin\pi\zeta},\tag{3.6}$$

$$\frac{n!}{\pi}\int_0^\infty I\left\{\frac{e^{i\psi}}{(s-e^{i\theta})^{n+1}}\right\}s^\zeta ds=\frac{(\zeta)_n\sin\left[(\pi-\theta)\zeta-(\psi-n\theta)\right]}{\sin\pi\zeta},\tag{3.7}$$

$$(-1)^{n+1}\frac{n!}{\pi}\int_0^\infty I\left\{\frac{e^{i\psi}}{(s+e^{i\theta})^{n+1}}\right\}s^\zeta ds=\frac{(\zeta)_n\sin\left[\theta\zeta+(\psi-n\theta)\right]}{\sin\pi\zeta}\tag{3.8}$$

where we set

$$(\zeta)_n=\zeta(\zeta-1)\cdots(\zeta-n+1).\tag{3.9}$$

Proof. (3.6) and (3.8) can be established for $-\pi<\theta<\pi$ by contour integration. (Alternatively, they can be deduced from known Mellin transforms. See [2, pp. 307–308].) Then (3.5) and (3.7) follow from (3.6) and (3.8) by substituting $\theta-\pi$ for θ.

Our first Abelian theorem is the following result.

THEOREM 3. *Suppose that* $u(z)$ *is in* H, *that* $-1<\delta<1$, *and that*

$$\alpha(t)=O(|t|^{1+\delta})$$

as $t\to 0$. *Then for all* $0<\theta<\pi$, ψ, *and* $n\geq 1$,

$$u(re^{i\theta})=O(r^\delta),\ \overset{n}{D}_\psi u(re^{i\theta})=O(r^{\delta-n})$$

as $r\to 0+$. *We can replace "O" by "o" in both the hypotheses and the conclusions.*

Proof. We will consider the "o" case. By hypothesis we can select $c>0$ such that

$$|\alpha(t)|\leq\epsilon|t|^{1+\delta}\tag{3.10}$$

for $|t|\leq c$ and by (3.4) we can assume that $\alpha(t)$ is constant outside of this interval. Next we can assume that $K=0$ in (3.1) and (3.2) and, with an integration by parts, we obtain

$$u(z)=\frac{1}{\pi}\int_{-\infty}^\infty I\left\{\frac{1}{(s-z)^2}\right\}\alpha(s)ds,$$

$$\overset{n}{D}_\psi u(z)=\frac{(n+1)!}{\pi}\int_{-\infty}^\infty I\left\{\frac{e^{i\psi}}{(s-z)^{n+2}}\right\}\alpha(s)ds.$$

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[1] When $\zeta=0$ in (3.5), (3.6), (3.7), (3.8), (4.12) and similarly when $\delta=0$ in (3.13), (4.5), (4.7), the left hand side of each equation is equal to the limit of the right hand side as the relevant variable, ζ or δ, approaches 0.

Then with (3.10) we have

$$|u(re^{i\theta})| \leq \frac{2r\sin\theta}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon|s|^{1+\delta}}{|s-re^{i\theta}|^3}ds,$$

$$|D_\psi^n u(re^{i\theta})| \leq \frac{(n+1)!}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon|s|^{1+\delta}}{|s-re^{i\theta}|^{n+2}}ds$$

and the desired conclusions follow immediately since, for $n\geq 1$,

$$\int_{-\infty}^{\infty}\frac{|s|^{1+\delta}}{|s-re^{i\theta}|^{n+2}}ds = r^{\delta-n}\int_{-\infty}^{\infty}\frac{|\sigma|^{1+\delta}}{|\sigma-e^{i\theta}|^{n+2}}d\sigma = Cr^{\delta-n}$$

where $C=C(\theta,\delta,n)<\infty$.

THEOREM 4. *Suppose that $u(z)$ is in H, that $-1<\delta<1$, and that*

$$\alpha(t)\sim At^{1+\delta},\quad -\alpha(-t)\sim Bt^{1+\delta} \tag{3.11}$$

(3.11)[2]

as $t\to 0+$. Then for all $0<\theta<\pi$, ψ, and $n\geq 1$,

$$u(re^{i\theta}) = u^*(re^{i\theta}) + o(r^\delta),\quad D_\psi^n u(re^{i\theta}) = D_\psi^n u^*(re^{i\theta}) + o(r^{\delta-n}) \tag{3.12}$$

as $r\to 0+$, where

$$u^*(re^{i\theta}) = (1+\delta)\frac{A\sin(\pi-\theta)\delta + B\sin\theta\delta}{\sin\pi\delta}r^\delta. \tag{3.13}$$

Proof. This last result follows almost directly from Theorem 3. First set

$$\alpha^*(t) = At^{1+\delta},\qquad -\alpha^*(-t) = Bt^{1+\delta} \tag{3.14}$$

for $0\leq t<\infty$; then $\alpha^*(t)$ satisfies (1.2) and

$$u^*(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} I\left\{\frac{1}{s-z}\right\}d\alpha^*(s)$$

is in H. From (3.11) and (3.14) it follows that

$$\alpha(t)-\alpha^*(t) = o(|t|^{1+\delta})$$

as $t\to 0$ and hence (3.12) follows from Theorem 3. Finally we see that $u^*(re^{i\theta})$ is simply

$$(1+\delta)\left\{\frac{A}{\pi}\int_0^\infty I\left(\frac{1}{\sigma-e^{i\theta}}\right)\sigma^\delta d\sigma - \frac{B}{\pi}\int_0^\infty I\left(\frac{1}{\sigma+e^{i\theta}}\right)\sigma^\delta d\sigma\right\}r^\delta$$

and (3.13) follows from (3.5) and (3.6) of Lemma 2 with $\zeta=\delta$.

[2] Here and in what follows $\phi(x)\sim Cx^p$ means $x^{-p}\phi(x)\to C$ as $x\to 0+$.

Theorem 4 establishes a relation between the behaviour of $\alpha(t)$ near $t=0$ and the behaviour of $u(z)$ near $z=0$ for $|\delta|<1$. (In particular we obtain the Fatou theorem when $\delta=0$ and $A=B$.) It is natural to inquire about the situation when $|\delta|=1$.

The following two results discuss the cases when $\delta=-1$ and $\delta=1$ respectively.

THEOREM 5. *Suppose that* $u(z)$ *is in* H *and that*

$$\alpha(t) - \alpha(-t) \to A\pi$$

as $t\to 0+$. *Then for all* $0<\theta<\pi$, ψ, *and* $n\geq 1$,

$$u(re^{i\theta}) = u^*(re^{i\theta}) + o(r^{-1}),$$
$$D_\psi^n u(re^{i\theta}) = D_\psi^n u^*(re^{i\theta}) + o(r^{-1-n})$$

as $r\to 0+$, *where* $u^*(x, y)=Ay(x^2+y^2)$.

THEOREM 6. *Suppose that* $u(z)$ *is in* H *and that the integral*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\alpha(s)}{s^2} = A \tag{3.15}$$

is absolutely convergent. Then for all $0<\theta<\pi$, ψ, *and* $n\geq 1$,

$$\begin{aligned} u(re^{i\theta}) &= u^*(re^{i\theta}) + o(r), \\ D_\psi^n u(re^{i\theta}) &= D_\psi^n u^*(re^{i\theta}) + o(r^{1-n}) \end{aligned} \tag{3.16}$$

as $r\to 0+$, *where* $u^*(x, y)=(A+K)y$.

Proofs. First note that Theorem 3 holds for $\delta=-1$; Theorem 5 is then established by an argument similar to the one used in the proof of Theorem 4.

The hypotheses of Theorem 6 imply that $1/s^2$ is integrable, in the Lebesgue-Stieltjes sense, over the real line. By (3.1),

$$\frac{u(re^{i\theta})}{r\sin\theta} = K + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\alpha(s)}{|s-re^{i\theta}|^2}.$$

For $s\neq 0$, $1/|(s-re^{i\theta})|^2$ is bounded by C/s^2 and converges to $1/s^2$ as $r\to 0+$, and the first part of (3.16) follows by the "dominated convergence" theorem. The second part follows similarly.

When $u(z)$ is in H, there is no significant Fatou theorem with $|\delta|>1$. For on the one hand, $\alpha(t)$ has left and right hand limits at each point and hence the conclusion of Theorem 5 *always* holds for some value of A. On the other hand, Theorem 6 shows us that for $\delta\geq 1$ the behaviour of $u(z)$ near $z=0$ depends upon the values $\alpha(t)$ assumes along the *entire* real line. For functions in H^+ this last fact is strikingly illustrated in Theorem 11.

4. **Extensions of the Loomis theorem.** We begin with a result which, for functions in H^+, is a converse for Theorem 3 (cf. [1, Lemma 1]).

THEOREM 7. *Suppose that $u(z)$ is in H^+, that δ is any real number, that $0<a<\pi$, and that for $\theta=a$,*

$$u(re^{i\theta}) = O(r^\delta) \tag{4.1}$$

as $r\to 0+$. Then (4.1) *holds for all $0<\theta<\pi$ and*

$$\alpha(t) = O(|t|^{1+\delta})$$

as $t\to 0$. We can replace "O" by "o" in the hypotheses and the conclusions.

Proof. It is not difficult to see that we may take $K=0$ in (3.1). Then, with

$$C = \frac{\sin\theta}{\sin a} \operatorname*{LUB}_{|\sigma|<\infty} \left|\frac{\sigma - e^{ia}}{\sigma - e^{i\theta}}\right|^2 < \infty,$$

we obtain

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{r\sin\theta}{|s-re^{i\theta}|^2}\, d\alpha(s) \\ &\le \frac{C}{\pi}\int_{-\infty}^{\infty} \frac{r\sin a}{|s-re^{ia}|^2}\, d\alpha(s) = Cu(re^{ia}). \end{aligned}$$

The second part follows since

$$\frac{\alpha(t)-\alpha(-t)}{t} \le 2\int_{-t}^{t} \frac{t}{s^2+t^2}\, d\alpha(s) \le 2\pi u(0, t)$$

for each $t>0$.

The next three theorems are converses for Theorem 4. In these results we assume that $0<a,\ b<\pi$, that $-1<\delta<1$, and that $m, n\ge 1$. Furthermore we set

$$\mu = c - ma, \qquad \eta = d - nb, \tag{4.2}$$

and we adopt the notation (3.9).

THEOREM 8. *Suppose that $u(z)$ is in H^+, that $a\ne b$, and that*

$$u(re^{ia}) \sim Ar^\delta,\quad u(re^{ib}) \sim Br^\delta \tag{4.3}$$

as $r\to 0+$. Then

$$\alpha(t) \sim Ct^{1+\delta},\quad -\alpha(-t) \sim Dt^{1+\delta} \tag{4.4}$$

as $t\to 0+$, where

$$\begin{aligned} (1+\delta)C &= \frac{A\sin b\delta - B\sin a\delta}{\sin(b-a)\delta}, \\ (1+\delta)D &= \frac{B\sin(\pi-a)\delta - A\sin(\pi-b)\delta}{\sin(b-a)\delta}. \end{aligned} \tag{4.5}$$

Theorem 8 was first proved by Allen and Kerr by a different method (see [1, Theorem III]).

THEOREM 9. *Suppose that* $u(z)$ *is in* H^+, *that* $(b-a)\delta+\eta\neq 0 \mod \pi$, *and that*

$$u(re^{ia}) \sim Ar^{\delta}, \qquad D_d^n(re^{ib}) \sim Br^{\delta-n} \tag{4.6}$$

as $r\to 0+$. *Then* (4.4) *holds with*

$$(1+\delta)C = \frac{(\delta)_n A \sin[b\delta+\eta] - B\sin a\delta}{(\delta)_n \sin[(b-a)\delta+\eta]}, \qquad (1+\delta)D = \frac{B\sin(\pi-a)\delta - (\delta)_n A\sin[(\pi-b)\delta-\eta]}{(\delta)_n \sin[(b-a)\delta+\eta]}. \tag{4.7}$$

THEOREM 10. *Suppose that* $u(z)$ *is in* H^+, *that* $(b-a)\delta+R-\mu\neq 0 \mod \pi$, *that* $\delta\neq 0$, *and that*

$$u(re^{ia}) = O(r^{\delta}), \qquad D_c^m u(re^{ia}) \sim Ar^{\delta-m}, \qquad D_d^n u(re^{ib}) \sim Br^{\delta-m}$$

as $r\to 0+$. *Then* (4.4) *holds with*

$$(1+\delta)C = \frac{(\delta)_n A\sin[b\delta+\eta] - (\delta)_m B\sin[a\delta+\mu]}{(\delta)_m(\delta)_n \sin[(b-a)\delta+\eta-\mu]},$$

$$(1+\delta)D = \frac{(\delta)_m B\sin[(\pi-a)\delta-\mu] - (\delta)_n A\sin[(\pi-b)\delta-\eta]}{(\delta)_m(\delta)_n \sin[(b-a)\delta+\eta-\mu]}.$$

Proofs. The proofs for these three theorems follow along similar lines. We give only the details for Theorem 9.

By (3.4) we can assume that $\alpha(t)$ is constant for large t; by (4.6) and Theorem 7 we see that

$$\alpha(t) = O(|t|^{1+\delta})$$

for small t. From these two observations it follows that

$$\beta(t) = \int_0^t \frac{d\alpha(s)}{s^{\delta}}, \qquad \gamma(t) = -\int_0^t \frac{d\alpha(-s)}{s^{\delta}} \tag{4.8}$$

are defined and that

$$\int_t^{et} \frac{|d\beta(s)|}{s} = \int_t^{et} \frac{d\alpha(s)}{s^{1+\delta}} \leq \frac{\alpha(et) - \alpha(t)}{t^{1+\delta}} = O(1),$$

$$\int_t^{et} \frac{|d\gamma(s)|}{s} = -\int_t^{et} \frac{d\alpha(-s)}{s^{1+\delta}} \leq \frac{\alpha(-t) - \alpha(-et)}{t^{1+\delta}} = O(1)$$

for all $t>0$. Hence $\beta(t)$ and $\gamma(t)$ are in V'. Furthermore, by a Tauberian theorem due to Allen and Kerr,

$$\frac{\alpha(t)}{t^{1+\delta}} \to C \quad \text{iff} \quad \frac{\beta(t)}{t} \to (1+\delta)C,$$
$$\frac{-\alpha(-t)}{t^{1+\delta}} \to D \quad \text{iff} \quad \frac{\gamma(t)}{t} \to (1+\delta)D \tag{4.9}$$

as $t \to 0+$ (see [1, Lemma 2]).

Once again we can assume that $K=0$ in (3.1) and (3.2). Then with (4.8) we obtain

$$u(z) = \frac{1}{\pi}\int_0^\infty I\left\{\frac{s^\delta}{s-z}\right\} d\beta(s) - \frac{1}{\pi}\int_0^\infty I\left\{\frac{s^\delta}{s+z}\right\} d\gamma(s),$$

$$D_\psi^n u(z) = \frac{n!}{\pi}\int_0^\infty I\left\{\frac{e^{i\psi}s^\delta}{(s-z)^{n+1}}\right\} d\beta(s) + (-1)^{n+1}\frac{n!}{\pi}\int_0^\infty I\left\{\frac{e^{i\psi}s^\delta}{(s+z)^{n+1}}\right\} d\gamma(s),$$

and with

$$f_1(t) = \frac{1}{\pi} I\left\{\frac{t^\delta}{t-e^{ia}}\right\}, \qquad f_2(t) = -\frac{1}{\pi} I\left\{\frac{t^\delta}{t+e^{ia}}\right\},$$
$$g_1(t) = \frac{n!}{\pi} I\left\{\frac{e^{id}t^\delta}{(t-e^{ib})^{n+1}}\right\}, \qquad g_2(t) = (-1)^{n+1}\frac{n!}{\pi} I\left\{\frac{e^{id}t^\delta}{(t+e^{ib})^{n+1}}\right\}, \tag{4.10}$$

we conclude that

$$r^{-\delta}u(re^{ia}) = \frac{1}{r}\int_0^\infty f_1\left(\frac{s}{r}\right) d\beta(s) + \frac{1}{r}\int_0^\infty f_2\left(\frac{s}{r}\right) d\gamma(s),$$
$$r^{n-\delta}D_d^n u(re^{ib}) = \frac{1}{r}\int_0^\infty g_1\left(\frac{s}{r}\right) d\beta(s) + \frac{1}{r}\int_0^\infty g_2\left(\frac{s}{r}\right) d\gamma(s) \tag{4.11}$$

for $r>0$. By virtue of (2.9), all the functions in (4.10) are in M' and, by Lemma 2 and (4.2), we have

$$F_1(y) = \frac{\sin(\pi - a)\zeta}{\sin \pi\zeta}, \qquad F_2(y) = \frac{\sin a\zeta}{\sin \pi\zeta},$$

$$G_1(y) = \frac{(\zeta)_n \sin[(\pi - b)\zeta - \eta]}{\sin \pi\zeta}, \qquad G_2(y) = \frac{(\zeta)_n \sin[b\zeta + \eta]}{\sin \pi\zeta},$$

where $\zeta=\delta+iy$. It follows that

$$F_1(y)G_2(y) - F_2(y)G_1(y) = \frac{(\zeta)_n \sin[(b-a)\zeta+\eta]}{\sin \pi\zeta} \neq 0 \tag{4.12}$$

for all real y and we appeal to (4.6), (4.11), Corollary 1, and (4.9) to complete the proof.

Unfortunately, the restriction that $\delta\neq 0$ is essential in Theorem 10. For consider

$$f(z) = (\log z)^i = e^{i \log (\log z)}$$

where, for $z=\rho e^{i\phi}$ and $0<\phi<\pi$, we define

$$\log z = \log \rho + i\phi.$$

Now $f(z)$ is analytic and $|f(z)|\leq 1$ in $y>0$. Furthermore it is not difficult to show that for all $0<\theta<\pi$ and $n\geq 1$,

$$\frac{d^n}{dz^n} f(re^{i\theta}) = o(r^{-n})$$

as $r\rightarrow 0+$. With $u=R(f)+1$ it follows that

$$\limsup_{r\rightarrow 0+} u(re^{i\theta}) - \liminf_{r\rightarrow 0+} u(re^{i\theta}) = 2e^{-\pi}, \tag{4.13}$$

for each $0<\theta<\pi$, and that $u(z)$ satisfies the hypotheses of Theorem 10 with $\delta=0$ and $A=B=0$. However, (4.4) cannot hold for any C and D for otherwise Theorem 4 would contradict (4.13).

Finally consider the situation when $|\delta|=1$. Though Theorem 5 has no converse we can invert Theorem 6 as follows (cf. [7, p. 208]).

THEOREM 11. *Suppose that $u(z)$ is in H^+ and that $0<a<\pi$. If*

$$u(re^{ia}) = O(r)$$

as $r\rightarrow 0+$, then the integral (3.15) is absolutely convergent. If

$$u(re^{ia}) = o(r) \tag{4.14}$$

as $r\rightarrow 0+$, then $u(z)\equiv 0$.

Proof. By Theorem 7 we see that

$$\frac{u(0,t)}{t} = K + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2+t^2} = O(1) \tag{4.15}$$

as $t\rightarrow 0+$ and, by Fatou's lemma, we conclude that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2} \leq \liminf_{t\rightarrow 0+} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2+t^2} < \infty. \tag{4.16}$$

With (4.14) we can replace "O" by "o" on the right side of (4.15) and we conclude that K and the left side of (4.16) are equal to zero; hence $\alpha(t)\equiv 0$.

5. **An application.** Here we use Theorem 9 to obtain an extension of the following result due to Lindelöf (see [6, p. 70]).

LINDELÖF THEOREM. *Suppose that $f(z)$ is analytic and bounded in $y>0$, that $0<a<\pi$, and that for $\theta=a$,*

$$f(re^{i\theta}) \rightarrow A + iB \tag{5.1}$$

as $r\rightarrow 0+$. Then (5.1) *holds for all $0<\theta<\pi$.*

LEMMA 3. *Suppose that $u(z)$ is in H, that $v(z)$ is a conjugate for $u(z)$, that $-1<\delta<1$, and that*

$$\alpha(t) = O(|t|^{1+\delta}) \tag{5.2}$$

as $t\rightarrow 0$. If $0<a<\pi$ and

$$u(re^{ia}) \sim Ar^{\delta} \tag{5.3}$$

as $r\rightarrow 0+$, then

$$\frac{\partial}{\partial r} u(re^{ia}) \sim \delta A r^{\delta-1}$$

as $r\rightarrow 0+$. If $0<b<\pi$ and

$$v(re^{ib}) \sim Br^{\delta}$$

as $r\rightarrow 0+$, then

$$\frac{\partial}{\partial r} v(re^{ib}) \sim \delta B r^{\delta-1}$$

as $r\rightarrow 0+$.

Proof. For the first part set $\phi(r)=u(re^{ia})$ for $r>0$. By (5.3), (5.2), and Theorem 3 we see that

$$\phi(r) \sim Ar^{\delta},$$
$$\phi''(r) = O(r^{\delta-2})$$

as $r\rightarrow 0+$, and we conclude that

$$\phi'(r) \sim \delta A r^{\delta-1}$$

as $r\rightarrow 0+$ by a well known Tauberian theorem due to Hardy and Littlewood (see [8, p. 194]). The second part follows similarly.

Observe that Lemma 3 allows us to deduce Theorem 8 from Theorem 9. For the hypotheses (4.3) imply that (5.2) holds, by virtue of Theorem 7, and hence that

$$u(re^{ia}) \sim Ar^{\delta},$$
$$D_b^1 u(re^{ib}) \sim \delta B r^{\delta-1}$$

as $r \to 0+$, by the first part of Lemma 3. Setting $d=b$ and $n=1$ in Theorem 9 completes the argument.

Now let $\mathfrak{A}^+$ denote the class of functions $f(z)$, analytic with $R(f) \geq 0$ in $y>0$, and assume that $0<a, b<\pi$ and that $-1<\delta<1$.

THEOREM 12. *Suppose that $f(z)$ is in $\mathfrak{A}^+$, that $(b-a)\delta \not\equiv (\pi/2) \bmod \pi$, and that*

$$R\{f(re^{ia})\} \sim Ar^{\delta}, \quad I\{f(re^{ib})\} \sim Br^{\delta} \tag{5.4}$$

as $r \to 0+$. Then for all $0<\theta<\pi$ and $n \geq 1$,

$$f(re^{i\theta}) = f^*(re^{i\theta}) + o(r^{\delta}), \quad \frac{d^n}{dz^n} f(re^{i\theta}) = \frac{d^n}{dz^n} f^*(re^{i\theta}) + o(r^{\delta-n}) \tag{5.5}$$

as $r \to 0+$, where

$$f^*(z) = \frac{Ae^{-ib\delta} + iBe^{-ia\delta}}{\cos (b-a)\delta} z^{\delta}. \tag{5.6}$$

Proof. Set $u=R(f)$ and $v=I(f)$; $u(z)$ is in H^+. By (5.4) we see that

$$u(re^{ia}) \sim Ar^{\delta}, \quad v(re^{ib}) \sim Br^{\delta} \tag{5.7}$$

as $r \to 0+$. With Theorem 7 and the second part of Lemma 3 we obtain

$$\frac{\partial}{\partial r} v(re^{ib}) \sim \delta B r^{\delta-1}$$

as $r \to 0+$, and appealing to the Cauchy-Riemann equations we conclude that

$$D_d^1 u(re^{ib}) \sim \delta B r^{\delta-1} \tag{5.8}$$

as $r \to 0+$, where $d=b-\pi/2$. Now apply Theorem 9 to the first part of (5.7) and to (5.8); we obtain (4.4) with

$$(1+\delta)C = \frac{A \cos b\delta + B \sin a\delta}{\cos (b-a)\delta},$$
$$(1+\delta)D = \frac{A \cos (\pi-b)\delta - B \sin (\pi-a)\delta}{\cos (b-a)\delta}.$$

From Theorem 4 it follows that for all $0<\theta<\pi$, ψ, and $n \geq 1$

$$
\begin{aligned}
u(re^{i\theta}) &= u^*(re^{i\theta}) + o(r^\delta), \\
D_\psi^n u(re^{i\theta}) &= D_\psi^n u^*(re^{i\theta}) + o(r^{\delta-n})
\end{aligned}
\tag{5.9}
$$

as $r \to 0+$, where $u^* = R(f^*)$ and $f^*(z)$ is as defined in (5.6). Set $v^* = I(f^*)$ and observe, by the Cauchy-Riemann equations and the second part of (5.9), that

$$
\frac{\partial}{\partial\phi} v(re^{i\phi}) = \frac{\partial}{\partial\phi} v^*(re^{i\phi}) + o(r^\delta)
\tag{5.10}
$$

as $r \to 0+$. We conclude(3) that for all $0 < \theta < \pi$,

$$
\begin{aligned}
v(re^{i\theta}) &= \int_b^\theta \frac{\partial}{\partial\phi} v(re^{i\phi}) d\phi + v(re^{ib}) \\
&= \int_b^\theta \frac{\partial}{\partial\phi} v^*(re^{i\phi}) d\phi + v^*(re^{ib}) + o(r^\delta) \\
&= v^*(re^{i\theta}) + o(r^\delta)
\end{aligned}
$$

as $r \to 0+$ and the proof is complete.

Finally if $|f(z)| \leq M$ we can replace $f(z)$ by $f(z) + M$ and obtain the Lindelöf theorem from Theorem 12 with $a = b$ and $\delta = 0$.

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(3) It is not difficult to show that (5.10) holds *uniformly* in ϕ for ϕ between b and θ.